Towards the Classification of Scalar Non-Polynomial Evolution Equations: Polynomiality in Top Three Derivatives

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Abstract

We prove that arbitrary (non-polynomial) scalar evolution equations of order $m \geq 7$, that are integrable in the sense of admitting the canonical conserved densities $\rho^{(1)}$, $\rho^{(2)}$, and $\rho^{(3)}$ introduced in [A.V. Mikhailov, A.B. Shabat and V.V Sokolov "The symmetry approach to the classification of integrable equations" in '*What is Integrability?* edited by V.E. Zakharov (Springer-Verlag, Berlin 1991)], are polynomial in the derivatives u_{m-i} for $i = 0, 1, 2$. We also introduce a grading in the algebra of polynomials in u_k with $k \geq m-2$ over the ring of functions in $x, t, u, \ldots, u_{m-3}$ and show that integrable equations are scale homogeneous with respect to this grading.

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1. INTRODUCTION

The classification problem for scalar integrable equations in one space dimension is solved in the work of Wang and Sanders [1] for the polynomial and scale invariant case where it is shown that integrable equations of order greater than or equal to seven are symmetries of third and fifth order equations. In subsequent papers these results are extended to the cases where negative powers are involved [2] but their methods were not applied to equations without polynomiality or scaling properties.

The aim of the present work is the study of the classification problem for arbitrary evolution equations. In our work, we use the "formal symmetry" method, where the existence of certain conserved densities is a necessary condition for integrability. Our main result is that evolution equations that are integrable in the sense above are polynomial in top three derivatives and have a certain scale homogeneity property.

The "formal symmetry" method introduced by Mikhailov, Shabat and Sokolov [3] have been used to obtain a preliminary classification of essentially non-linear third order equations and quasi-linear fifth order equations. The classification of essentially non-linear equations is considered later in the work of Svinolupov [4], where quasi-linear integrable equations that are not linearizable are found to be related to the Korteweg-deVries and Krichever-Novikov equations through differential substitutions.

The first result towards a classification for arbitrary m'th order evolution equations is obtained in [5] where it is shown that scalar evolution equations $u_t = F[u]$, of order $m = 2k+1$ with $m \geq 7$, admitting a nontrivial conserved density $\rho = Pu_n^2 + Qu_n + R$ of order $n = m+1$, are quasi-linear. The method of [5] is not applicable to third order equations, because for $m = 3$, the canonical conserved density $\rho^{(1)}$ is not of the generic form on which the quasi-linearity result

is based on. For $m = 5$, although the generic form of $\rho^{(1)}$ is valid, $k = 2$ occurs as an exception and we cannot exclude the existence of fifth order non-quasi-linear integrable equations, at least with the present method. For $m \geq 7$ the structure of integrable equations seem to be much simpler and one may hope to obtain a complete classification as in the polynomial case.

In the present paper we continue with the classification problem using formal symmetries and we prove that evolution equations of order $m = 2k + 1 \geq 7$ admitting the canonical densities $\rho^{(i)}$, $i = 1, 2, 3$, as given in Appendix A, are polynomial in the derivatives u_m, u_{m-1} and u_{m-2}. The final result presented in Corollary 4.6.2 gives the explicit form of a candidate for an integrable evolution equation as a polynomial in u_m, u_{m-1} and u_{m-2}, with coefficients as yet undetermined functions of lower order derivatives. This result is definitely the best one can obtain by the use of the canonical densities $\rho^{(i)}$, for i up to 3, as discussed Remark 4.7 and in the conclusion. Any further progress towards polynomiality would require the computation of new canonical densities. These computations for general m are extremely tedious and they are deferred to future work. There is also an alternative and more promising direction towards the classification: Our candidates for integrable equations have a certain scale homogeneity property with respect to a grading in the algebra of polynomials in u_k with $k \geq m - 2$ over the ring of functions in $x, t, u, \ldots, u_{m-3}$. This grading called the "level grading" and introduced in [8] proved to be an efficient tool for practical computations and it is expected to allow the treatment of the polynomiality in lower orders in a unified manner. However, as both directions of approach to the problem require completely different techniques, the present paper is limited to the information that can be extracted from the existence of the conserved densities $\rho^{(i)}$, for $i \leq 3$.

The notation and terminology are reviewed in Section 2 and classification results are given

in Sections 3 and 4, where we show that for sufficiently large m, integrable equations of order m are polynomial in the top 3 derivatives, u_m, u_{m-1} and u_{m-2}. These results are obtained from the requirement that the canonical densities $\rho^{(i)}$, $i = 1, 2, 3$, be conserved quantities. The expression of these canonical densities are given in Appendix A. In Appendix B, we present general formulas for the k'th order derivatives of differential functions for large k. Due to the restrictions on the validity of these derivatives, the general formulas obtained in Section 4 are valid for $m \geq 19$. For $m < 19$, we have done explicit computations with the symbolic programming language REDUCE and obtained the same polynomiality results, as summarized in Section 5.

2. NOTATION AND TERMINOLOGY

Let $u = u(x,t)$. A function φ of x, t, u and the derivatives of u up to a fixed but finite order will be called a "differential function" [6] and denoted by $\varphi[u]$. We shall assume that φ has partial derivatives of all orders. We shall denote indices by subscripts or superscripts in parentheses such as in $\alpha_{(i)}$ or $\rho^{(i)}$ and reserve subscripts without parentheses for partial derivatives. For example if $u = u(x,t)$, then

$$u_t = \frac{\partial u}{\partial t}, \quad u_k = \frac{\partial^k u}{\partial x^k},$$

while for $\varphi = \varphi(x, t, u, u_1, \ldots, u_n)$,

$$\varphi_t = \frac{\partial \varphi}{\partial t}, \quad \varphi_x = \frac{\partial \varphi}{\partial x}, \quad \varphi_k = \frac{\partial \varphi}{\partial u_k}.$$

For φ as above, the total derivative with respect to x is denoted by $D\varphi$ and it is given by

$$D\varphi = \sum_{i=0}^{n} \varphi_i u_{i+1} + \varphi_x. \tag{2.1}$$

Higher order derivatives can be computed by applying the binomial formula

$$D^k\varphi = \sum_{i=0}^{n}\left[\sum_{j=0}^{k-1}\binom{k-1}{j}\left(D^j\varphi_i\right)u_{i+k-j}\right] + D^{k-1}\varphi_x. \tag{2.2}$$

The total derivative with respect to time denoted by D_t is given by

$$D_t\varphi = \sum_{i=0}^{n}\varphi_i D^i F \ + \ \varphi_t. \tag{2.3}$$

We recall that a differential function ρ is called a conserved density, provided that there is a differential function σ such that $D_t\rho = D\sigma$. If ρ is polynomial in certain higher derivatives, in order to check the conserved density condition, one can proceed with integration by parts and require the vanishing of the terms that are nonlinear in these highest derivatives. In our derivations we shall use only the vanishing of the coefficients of top two nonlinearities. As shown in Proposition 3.1, these terms come from top 4 derivatives in the expansion of $D_t\rho$. The general expression for $D^k\varphi$ given by $(A.6d)$ is valid for $k \geq 7$ and as we use the derivative $D^{k-2}F$ in the general formulas, we need $k \geq 9$ hence $m \geq 19$ for the validity of the general expressions.

We shall denote generic functions φ that depend on at most u_n by $O(u_n)$ or by $|\varphi| = n$. That is

$$\varphi = O(u_n) \quad \text{or} \quad |\varphi| = n \quad \text{if and only if} \quad \frac{\partial\varphi}{\partial u_{n+k}} = 0 \quad \text{for} \quad k \geq 1.$$

If $\varphi = O(u_n)$, then $D\varphi$ is linear in u_{n+1} and $D^k\varphi$ is polynomial in u_{n+i} for $i \geq 1$. In order to distinguish polynomial functions we use the notation $\varphi = P(u_n)$, i.e.,

$$\varphi = P(u_n) \quad \text{if and only if} \quad \varphi = O(u_n) \quad \text{and} \quad \frac{\partial^k\varphi}{\partial u_n^k} = 0 \quad \text{for some } k.$$

This distinction have been used in the expression of derivatives given in Appendix B.

Note that the total derivative with respect to x increases the order by one, thus if $|\varphi| = n$ then $|D^k\varphi| = n + k$. Furthermore, when $u_t = F$, with $|F| = m$, D_t increases the order by m.

Equalities up to total derivatives with respect to x will be denoted by $\cong$, i.e.,

$$\varphi \cong \psi \quad \text{if and only if} \quad \varphi = \psi + D\eta$$

We shall repeatedly use integration by parts of the following type of expressions.

Let $p_1 < p_2 < \ldots < p_l < s-1$ and $|\varphi| = k < p_1$. Then

$$\begin{aligned} \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s &\cong -D\left(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\right) u_{s-1}, \\ \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_{s-1}^{p} u_s &\cong -\tfrac{1}{p+1} D\left(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\right) u_{s-1}^{p+1}. \end{aligned} \tag{2.4}$$

These integrations by parts are repeated until one encounters a monomial which is nonlinear in the highest derivative.

The order of a differential monomial is not invariant under integration by parts, but one can easily compute the order of the nonlinear term that will result after integrations by parts.

3. GENERAL RESULTS ON CLASSIFICATION

We start by a brief description of the formal symmetry method [3]. If R is a recursion operator for the evolution equation $u_t = F[u]$, then any fractional power of R is also a recursion operator. Thus starting from a recursion operator of order 1, expanded in a formal series in inverse powers of D, one can compute the operators R^k of orders k, for $k = 2, 3, \ldots$. Each of these operators satisfy the operator equation

$$R_t^k + [R^k, F_*] = 0,$$

where F_* is the Frechet derivative of F, defined by $F_* = \sum_{i=0}^{m} F_i D^i$. It is known that for any two formal series A and B, the coefficient of D^{-1} is a total derivative [7], hence the coefficient of D^{-1} in each R^k, denoted by $\rho^{(k)}$, is a conserved quantity. In addition to these, for any m, it

is known that

$$\rho^{(-1)} = F_m^{-1/m} \quad \text{and} \quad \rho^{(0)} = F_{m-1}/F_m$$

are conserved densities. We recall that as stated in Section 2, subscripts denote differentiation with respect to the derivatives of u, that is $F_m = \frac{\partial F}{\partial u_m}$ and $\rho_{m,n}$ denotes $\frac{\partial^2 \rho}{\partial u_m u_n}$, etc.

The explicit expressions of $\rho^{(1)}$ and $\rho^{(2)}$ for $m \geq 5$ and of $\rho^{(3)}$ for $m \geq 7$ obtained in [5] are given in Appendix A.

In order to compute $D_t\rho$ up to total derivatives, we use (2.3) and integrate by parts until we encounter a term which is nonlinear in the highest derivative. The derivation of the polynomiality result is based on the coefficients of top two nonlinear terms, given by the equations (3.2) and (3.3), in Proposition 3.3. To arrive to these expressions we first show in Proposition 3.1 that if F has order $m = 2k+1$ and ρ has order $n = m + l$, then, for $m \geq 19$, the top two nonlinear terms are u_{3k+l+1}^2 and u_{3k+l}^2.

Proposition 3.1 *Let $\rho = \rho(x, t, u, \dots, u_n)$ be a differential function of order n and $u_t = F(x, t, u, \dots, u_m)$ be an evolution equation of order m, where $m = 2k+1$, $n = m + l$ and $k + l - 1 \geq 0$. Then, up to total derivatives $D_t\rho$ is*

$$\begin{aligned}(-1)^{k+1} D_t\rho \;\; &\cong \;\; \left(D^{k+1}\rho_n - D^k\rho_{n-1}\right) D^{k+l}F - \left(D^k\rho_{n-2} - D^{k-1}\rho_{n-3}\right) D^{k+l-1}F \\ &\quad +\varphi. \qquad (3.1)\end{aligned}$$

where $\varphi = O(u_{3k+l-1})$.

Proof. In $D_t\rho = \sum_{i=0}^n \rho_i D^i F + \rho_t$, the highest order derivative comes from $\rho_n D^n F$, where ρ_n and $D^n F$ are of orders $2k+l+1$ and $4k+l+2$ respectively. If we integrate by parts $k+1$ times we obtain

$$\rho_n D^n F \cong (-1)^{k+1} D^{k+1}\rho_n \; D^{k+l}F,$$

where $D^{k+1}\rho_n$ and $D^{k+l}F$ are now respectively of orders $3k+l+2$ and $3k+l+1$. One more integration by parts gives a term nonlinear in u_{3k+l+1}. Similarly one can see that in $\rho_{n-1}D^{n-1}F$, ρ_{n-1} and $D^{n-1}F$ are of orders $2k+l+1$ and $4k+l+1$. This time, integrating by parts k times, we have

$$\rho_{n-1}D^{n-1}F \cong (-1)^k D^k\rho_n \; D^{k+l}F,$$

where $D^k\rho_{n-1}$ and $D^{k+l}F$ are both of orders $3k+l+1$. Thus the highest order nonlinear term in u_{3k+l+1} comes from top two derivatives in $\rho_n D^n$ and $\rho_{n-1}D^{n-1}F$. By similar counting arguments, one can easily see that top two nonlinear terms are obtained from top four derivatives and the remaining terms are of order $3k+l-1$. $\square$

Remark 3.2 As the general expressions for the derivatives given in (A.6a-d) are valid for large k, there are restrictions on the validity of the formula (3.1). Since the top four terms of $D^{k+l}F$ and $D^{k+1}\rho_n$ are needed in (3.1), from (A.6d) it follows that $k+1$ and $k+l$ should be both larger than or equal to 7. On the other hand, at most top two terms of the expressions in the second bracket in (3.1) contribute to the top nonlinearities and it turns out that the restrictions coming from (A.6a,b) are always satisfied and the crucial restriction is $k+1 \geq 7$ and $k+l \geq 7$. Thus for $l = 1,\ 0,\ -1$ and -2, we need respectively $k \geq 6 (m \geq 13)$, $k \geq 7 (m \geq 15)$, $k \geq 8 (m \geq 17)$ and $k \geq 9 (m \geq 19)$.

We shall now give the explicit expressions of the coefficients of top two nonlinear terms for $m \geq 19$ and $k+l \geq 7$.

Proposition 3.3 *Let $\rho = \rho(x,t,u,\ldots,u_n)$ be a differential function of order n and $u_t = F(x,t,u,\ldots,u_m)$ be an evolution equation of order m, where $m = 2k+1$,$n = m+l$. Then for $k \geq 9$ and $k+l \geq 7$, the coefficients of the top two nonlinear terms u_{3k+l+1}^2 and u_{3k+l}^2 in the*

expression of $D_t\rho$ up to total derivatives is

$$(-1)^{k+1} D_t \rho \cong \Lambda_1 \; u_{3k+l+1}^2 + \Lambda_0 \; u_{3k+l}^2 + \varphi$$

where $\varphi = O(u_{3k+l-1})$ and Λ_1 and Λ_0 given below.

$$\Lambda_1 = (k+\tfrac{1}{2}) F_m D\rho_{n,n} - (k+l+\tfrac{1}{2}) DF_m \rho_{n,n} - F_{m-1}\rho_{n,n}, \tag{3.2}$$

$$\begin{aligned}
\Lambda_1 \;=\;& \rho_{n,n}\; D^3F_m \;\left[\tfrac{1}{12}\;(2k^3+6k^2l+6kl^2+2l^3+3k^2+3l^2+6kl+k+l)\right] \\
+\;& \rho_{n,n}\; D^2F_{m-1} \;\left[\tfrac{1}{2}\;(k^2+2kl+2k+2l+l^2+1)\right] \\
+\;& \rho_{n,n}\; DF_{m-2} \;\left[\tfrac{1}{2}\;(3+2k+2l)\right] \\
+\;& \rho_{n,n}\; F_{m-3} \\
+\;& D\rho_{n,n}\; D^2F_m \;\left[\tfrac{1}{4}\;(-2k^3-4k^2l-2kl^2+k^2+l^2+2kl+k+l)\right] \\
+\;& D\rho_{n,n}\; DF_{m-1} \;\left[\tfrac{1}{2}\;(1+l-2k^2-2kl)\right] \\
+\;& D\rho_{n,n}\; F_{m-2} \;\left[\tfrac{1}{2}\;(1-2k)\right] \\
+\;& D^2\rho_{n,n}\; DF_m \;\left[\tfrac{1}{4}(2k^3+2k^2l-k^2-k)\right] \\
+\;& D^2\rho_{n,n}\; F_{m-1} \;\left[\tfrac{1}{2}\;k^2\right] \\
+\;& D^3\rho_{n,n}\; F_m \;\left[\tfrac{1}{12}\;(-2k^3-3k^2-k)\right] \\
+\;& D\rho_{n,n-1}\; DF_m \;\left[\tfrac{1}{2}\;(-1+2k+2l)\right] \\
+\;& D\rho_{n,n-1}\; F_{m-1} \\
+\;& D^2\rho_{n,n-1}\; F_m \;\left[\tfrac{1}{2}\;(-1-2k)\right] \\
+\;& \rho_{n,n-2}\; DF_m \;[2k+2l-1] \\
+\;& \rho_{n,n-2}\; F_{m-1} \;[2]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad D\rho_{n,n-2}\ F_m\ \ [-2k-1] \\
&+ \quad \rho_{n-1,n-1}\ DF_m\ \ \left[\tfrac{1}{2}\ (1-2k-2l)\right] \\
&+ \quad \rho_{n-1,n-1}\ F_{m-1}\ \ [-1] \\
&+ \quad D\rho_{n-1,n-1}\ F_m\ \ \left[\tfrac{1}{2}\ (1+2k)\right].
\end{aligned} \tag{3.3}
$$

Proof. The proof is a straightforward computation of the integrations indicated in Proposition 3.1. Writing the first four terms in $D_t\rho$ and keeping only the terms which contribute to the nonlinearities $u_{3k+l+1}^2,\ \ u_{3k+l}^2$, we get

$$
\begin{aligned}
(-1)^{k+1}D_t\rho \ \ \cong\ \ & \rho_{n,n}\ F_m u_{3k+l+1}\ u_{3k+l+2} \\
+ \quad & \rho_{n,n}\ [F_{m-1}+(k+l)DF_m]\ u_{3k+l}\ u_{3k+l+2} \\
+ \quad & \rho_{n,n}\ \left[F_{m-2}+(k+l)DF_{m-1}+\tbinom{k+l}{2}D^2F_m\right]\ u_{3k+l-1}\ u_{3k+l+2} \\
+ \quad & \rho_{n,n}\ \left[F_{m-3}+(k+l)DF_{m-2}+\tbinom{k+l}{2}D^2F_{m-1}+\tbinom{k+l}{3}D^3F_m\right]\ u_{3k+l-2}\ u_{3k+l+2} \\
+ \quad & (k+1)\ D\rho_{n,n}\ F_m\ u_{3k+l+1}\ u_{3k+l+1} \\
+ \quad & (k+1)\ D\rho_{n,n}\ [F_{m-1}+(k+l)DF_m]\ u_{3k+l}\ u_{3k+l+1} \\
+ \quad & (k+1)\ D\rho_{n,n}\ \left[F_{m-2}+(k+l)DF_{m-1}+\tbinom{k+l}{2}D^2F_m\right]\ u_{3k+l-1}\ u_{3k+l+1} \\
+ \quad & \left[\rho_{n,n-2}+D\rho_{n,n-1}+\tbinom{k+1}{2}D^2\rho_{n,n}-\rho_{n-1,n-1}\right]\ F_m u_{3k+l+1}\ u_{3k+l} \\
+ \quad & \left[\rho_{n,n-2}+D\rho_{n,n-1}+\tbinom{k+1}{2}D^2\rho_{n,n}-\rho_{n-1,n-1}\right] \\
& \qquad \times [F_{m-1}+(k+l)DF_m]\ u_{3k+l}\ u_{3k+l} \\
+ \quad & \left[\rho_{n,n-3}+(k+1)D\rho_{n,n-2}+kD^2\rho_{n,n-1}+\tbinom{k+1}{3}D^3\rho_{n,n}-\rho_{n-1,n-2}-kD\rho_{n-1,n-1}\right] \\
& \qquad \times F_m\ u_{3k+l+1}\ u_{3k+l-1} \\
- \quad & \rho_{n-2,n}\ F_m\ u_{3k+l}\ u_{3k+l+1} \\
- \quad & \rho_{n-2,n}\ [F_{m-1}+(k+l-1)DF_m]\ u_{3k+l-1}\ u_{3k+l+1}
\end{aligned}
$$

$$- \quad [\rho_{n-2,n-1} + kD\rho_{n-2,n} - \rho_{n-3,n}]\ F_m\ u_{3k+l}\ u_{3k+l}. \tag{3.4}$$

After integrations by parts we get (3.2) as the coefficient of the first nonlinear term u_{3k+l+1}^2 and (3.3) as the coefficient of the second nonlinear term u_{3k+l}^2. □

Actually the expression of Λ_1 is valid for $m \geq 13$ and $k + l \geq 7$. It is clear that if ρ is a conserved density for the evolution equation $u_t = F$, then necessarily one should have

$$\Lambda_1 = 0, \qquad \Lambda_0 = 0.$$

These two equations will be used repeatedly in order to derive a number of necessary condition for integrability.

From equation (3.2) we can easily get a number of results pertaining the form of the conserved densities. In particular we can see that higher order conserved densities should be quadratic in the highest derivative and top coefficients of the conserved densities at every order are proportional to each other [5].

Corollary 3.4 *Let $\rho = \rho(x, t, u, \dots, u_n)$ be a differential function of order n and $u_t = F(x, t, u, \dots, u_m)$, be an evolution equation of order m, $m \geq 7$ and $n > m$. Then*

$$\rho_{n,n,n} = 0 \tag{3.5}$$

Proof. It can be seen that (3.2) uses only top two terms and for $l > 0$ it is valid for $k + 1 \geq 3$. Writing it in the form

$$(k + \tfrac{1}{2})\frac{D\rho_{n,n}}{\rho_{n,n}} - (k + l + \tfrac{1}{2})\frac{DF_m}{F_m} = \frac{F_{m-1}}{F_m}, \tag{3.6}$$

we can see that for $n > m$ the highest order term is $D\rho_{n,n}$ and it follows that $\rho_{n,n,n} = 0$. □

Remark 3.5 From (3.6) one can easily see that if ρ and η are both conserved densities of order n, with $\rho_{n,n} = P$ and $\eta_{n,n} = Q$, then $\frac{DP}{P} = \frac{DQ}{Q}$, hence the ratio of the top coefficients

is independent of x. If ρ and η are conserved densities of consecutive orders say, $|\rho| = n$ and $|\eta| = n + 1$ with $\rho_{n,n} = P$ and $\eta_{n+1,n+1} = Q$, then

$$(k + \tfrac{1}{2})\left(\frac{DQ}{Q} - \frac{DP}{P}\right) = \frac{DF_m}{F_m},$$

hence $Q = F_m^{2/m} P$.

Remark 3.6 If the partial derivatives of F and ρ in (3.2) and (3.3) depend at most on u_j, then these equations are polynomial in u_{j+i}, $i > 0$. In all the subsequent computations we have used only the coefficient of the top order derivatives.

4. POLYNOMIALITY RESULTS FOR THE GENERAL CASE

In this section we shall obtain polynomiality results, applying Proposition 3.3 either directly to a canonical density $\rho^{(i)}$, $i = 1, 2, 3$ (step 3 and 6), or to generic conserved densities ρ, ν and show that one of the canonical densities is of that generic form (see Table 1). These derivations involve quite complicated and technical derivations organized in 6 steps. The intermediate results displaying polynomiality in u_m, u_{m-1} and u_{m-2} are presented respectively in Corollaries 4.1.2, 4.3.2 and 4.6.2, labeled as "Result A", "Result B" and "Result C".

Although the method seems to be a straightforward application of the equations (3.2) and (3.3) the computations are tedious and it took a number of trial and errors to discover the right sequence of computations presented as Steps 1 through 6. The computations are done in part analytically, in part with the symbolic programming language REDUCE. As the dependencies on lower order derivatives become explicit, the size of the actual expressions of the canonical densities grow very fast and we had to use generic forms for these conserved densities whenever possible. At each step, we tried to deduce from equations (3.2) and (3.3) a homogeneous linear system of equations, whose non-singularity leads to a polynomiality result. It is a remarkable

fact in many cases these systems become singular for $k = 2$, hence our results are not applicable to fifth order equations.

We use generic conserved densities in steps one, two and four. The first step is to obtain the quasi-linearity result for $m > 5$, which follows from the fact that the coefficient matrix of a homogeneous system is non-singular for $m > 5$. At the second and fourth steps we have a similar structure; we show that the coefficient of u_m is independent of u_{m-1} and u_{m-2} respectively, by obtaining nonsingular homogeneous systems of linear equations.

The third and sixth steps are based on relatively straightforward computations using the canonical densities. At the third step we complete polynomiality in u_{m-1} while at the fifth and sixth steps we complete polynomiality in u_{m-2}, by using the explicit form of the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$.

4.1. Step 1. quasi-linearity: $F_{m,m} = 0$

We now prove that an evolution equation admitting a quadratic conserved density is quasi-linear. The quasi-linearity result was already given in [5], but we shall repeat this derivation, because with Proposition 3.3, the proof given below is much shorter and neater.

Proposition 4.1.1 *Let* $u_t = F(x, t, u, \ldots, u_m)$, *with* $m = 2k + 1 \geq 13$ *and assume that it admits a conserved density of the form*

$$\rho = P u_{m+1}^2 + Q u_{m+1} + R \tag{4.1}$$

where $|P| = |Q| = |R| = m$. *Then* $P F_{m,m} = 0$.

Proof. Since the conserved density has order $m+1$, the restriction for the applicability of (A.6d) is $k + 1 \geq 7$ hence $m \geq 13$. The coefficient of u_{m+1} in (3.2) and the coefficient of u_{m+3} in (3.3)

are respectively as follows:

$$(2k+1)\,F_m P_m - (2k+3)\,PF_{m,m} = 0 \tag{4.2}$$

$$(2k+1)(k^2+k+6)F_m P_m - (2k+3)(k+1)(k+2)PF_{m,m} = 0 \tag{4.3}$$

These two equations form the following homogeneous system of linear equations

$$\begin{bmatrix} 1 & -1 \\ (k^2+k+6) & -(k^2+3k+2) \end{bmatrix} \begin{bmatrix} F_m P_m \\ PF_{m,m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.4}$$

The coefficient matrix in this linear system is singular for $k = 2$. For $k \neq 2$ the homogeneous system has only the trivial solution hence from (4.4) we conclude that

$$F_m P_m = PF_{m,m} = 0. \tag{4.5}$$

□

The quasi-linearity result follows from here immediately.

Result A: Polynomiality in u_m

Corollary 4.1.2 *If the canonical density $\rho^{(1)}$ is a conserved quantity for the evolution equation $u_t = F$, then $F_{m,m} = 0$, hence*

$$u_t = Au_m + B$$

where A, B are functions of $x, t, u, \ldots u_{m-1}$.

Proof. $\rho^{(1)}$ in $(A.3)$ is of the form (4.1) with $P = a^{-1}a_m^2$ where $a = F_m^{1/m}$. Hence $PF_{m,m} = 0$ implies $F_{m,m} = 0$, and $u_t = Au_m + B$ □

In reference [5], by direct computation of the conserved density conditions, it was shown that equations of orders 7, 9 and 11 were also quasi-linear. Since the existence of the canonical

densities are necessary conditions for integrability, it follows that integrable evolution equations of order $m = 2k+1$ with $k \geq 3$ are quasi-linear. As discussed in Proposition 3.1, the applicability of the general formulas to the computation of the conserved density conditions require that $k + l \geq 7$, hence the results below are valid only for $m \geq 19$. For lower orders we have checked the validity of our results by direct computations and outline the results in Section 5.

4.2 Step 2: Polynomiality in u_{m-1}, first result, $A_{m-1} = 0$

In the second and third steps we determine the dependency of the coefficients A and B, in $u_t = Au_m + B$ on u_{m-1}. For this purpose we consider a generic quadratic conserved density ρ of order m and use the coefficients of the top two nonlinearities in $D_t\rho$ which are respectively given by equations (3.2) and (3.3).

Proposition 4.2.1 *Let* $u_t = Au_m + B, \;\; m \geq 19$ *with* $|A| = |B| = m - 1$, *and assume that it admits a conserved density*

$$\rho = Pu_m^2 + Qu_m + R, \tag{4.6}$$

where $|P| = |Q| = |R| = m - 1$. *Then*

$$PA_{m-1} = 0. \tag{4.7}$$

Proof. We use here the equations (3.2) and (3.3) with $l = 0, \;\; F_m = A$ and $\rho_{n,n} = 2P$. The coefficient of u_m in (3.2) is

$$(2k + 1)\, P_{m-1}A - (2k + 3)\, A_{m-1}P = 0, \tag{4.8}$$

while the coefficient of u_{m+2} in (3.3) is

$$\left[2k^3 + 9k^2 + 13k + 6\right] A_{m-1}P - \left[2k^3 + 3k^2 + 13k + 6\right] P_{m-1}A = 0. \tag{4.9}$$

From (4.8) and (4.9) we get the following linear system.

$$\begin{bmatrix} 2k+1 & -(2k+3) \\ 2k^3+3k^2+13k+6 & -(2k^3+9k^2+13k+6) \end{bmatrix} \begin{bmatrix} AP_{m-1} \\ PA_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.10}$$

The coefficient matrix of the linear system is singular for $k=2$, then $m=5$ which appear as an exception. For $k \neq 2$, the system has a trivial solution. Hence $PA_{m-1} = AP_{m-1} = 0$. $\square$

Again by using the canonical density $\rho^{(1)}$ we shall obtain that A is independent of u_{m-1}

Corollary 4.2.2 *If the canonical density $\rho^{(1)}$ is conserved, then $A_{m-1} = 0$.*

Proof. We substitute $u_t = Au_m + B$ in $(A.3)$ and integrate by parts we can see that $\rho^{(1)}$ is of the form (4.6) where

$$P = \frac{a_{m-1}^2}{a}, \qquad A = a^m. \tag{4.11}$$

Hence $PA_{m-1} = 0$ implies $A_{m-1} = 0$. $\square$

4.3 Step 3: Polinomiality in u_{m-1}, second result, $B_{m-1,m-1,m-1} = 0$

Now we shall see that the conservation of the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$ will determine the form of B. By substituting $u_t = Au_m + B$, with $|A| = m-2$ and $|B| = m-1$ in the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$ one can see that they reduce to

$$\rho^{(1)} = P^{(1)}u_{m-1}^2 + Q^{(1)}u_{m-1} + R^{(1)}$$

$$\rho^{(3)} = P^{(3)}u_{m}^2 + Q^{(3)}u_{m} + R^{(3)}$$

where $|Q^{(1)}| = |R^{(1)}| = m-2$, and $|Q^{(3)}| = |R^{(3)}| = m-1$, and

$$P^{(1)} = \frac{24}{m^2-1}a_{m-2,m-2} + a_{m-2}^2 a^{-1}(m^2-1), \tag{4.12}$$

$$
\begin{aligned}
P^{(3)} &= \frac{a}{m^3+3m^2-m-3}\left[a_{m-2}^2\left(m^3+3m^2-121m+597\right)\right. \\
&+ 60a^{-m+1}a_{m-2}B_{m-1,m-1}\left(\tfrac{3}{m}-1\right) \\
&+ \left. 60a^{-2m+2}B_{m-1,m-1}^2\left(\tfrac{1}{m}-\tfrac{1}{m^2}\right)\right] \qquad (4.13)
\end{aligned}
$$

Proposition 4.3.1 *Let* $u_t = Au_m + B,\ \ m \geq 19$ *with* $|A| = m-2$ *and* $|B| = m-1$. *Then if the canonical densities* $\rho^{(1)}$ *and* $\rho^{(3)}$ *are conserved quantities, then*

$$
B_{m-1,m-1,m-1} = 0. \qquad (4.14)
$$

Proof. To prove this result, it would be actually sufficient to use $\rho^{(3)}$ only, but we first see how far one can go by using $\rho^{(1)}$. We first compute (3.2) for $\rho = \rho^{(1)}$, hence with $l = -1,\ \ F_m = A$ and $\rho_{n,n} = \rho_{m-1,m-1} = 2P^{(1)}$ to get

$$
(k+\tfrac{1}{2})AP^{(1)}_{m-2}u_{m-1} - (k-\tfrac{1}{2})P^{(1)}A_{m-2}u_{m-1} - P^{(1)}B_{m-1} = 0. \qquad (4.15)
$$

Differentiating (4.15) twice with respect to u_{m-1} we obtain

$$
2P^{(1)}B_{m-1,m-1,m-1} = 0. \qquad (4.16)
$$

If $P^{(1)}$ is nonzero, then B is quadratic in u_{m-1}, but $P^{(1)} = 0$ gives a differential equation for a and we cannot exclude the possibility that $B_{m-1,m-1,m-1} \neq 0$. It is possible to solve this differential equation, but it is easier to use $\rho^{(3)}$.

Now we compute (3.2) using $\rho = \rho^{(3)}$, hence with $l = 0,\ \ F_m = A$ and $\rho_{n,n} = \rho_{m,m} = 2P^{(3)}$ to get

$$
(k+\tfrac{1}{2})AP^{(3)}_{m-1}u_m - (k+\tfrac{1}{2})P^{(3)}A_{m-2}u_{m-1} - P^{(3)}B_{m-1} = 0. \qquad (4.17)
$$

This equation is linear in u_m and its coefficient gives

$$
(k+\tfrac{1}{2})AP^{(3)}_{m-1} = 0. \qquad (4.18)
$$

Now since A should be nonzero, it follows that $P^{(3)}$ is independent of u_{m-1}. Differentiating $P^{(3)}$, in (4.13), with respect to u_{m-1}, we obtain

$$B_{m-1,m-1,m-1}\left[a^{-m+1}a_{m-2}\left(\frac{3}{m}-1\right)+2a^{-2m+2}B_{m-1,m-1}\left(\frac{1}{m}-\frac{1}{m^2}\right)\right]=0. \tag{4.19}$$

Thus we can conclude that:

$$B_{m-1,m-1,m-1}=0. \tag{4.20}$$

Result B: Polynomiality in u_m, u_{m-1}

Corollary 4.3.2 *If the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$ are conserved quantities for the evolution equation $u_t = F$, then*

$$F = Au_m + Cu_{m-1}^2 + Du_{m-1} + E. \tag{4.21}$$

where A, C, D, E are functions of $x, t, u, \ldots u_{m-2}$.

4.4 Step 4: Polinomiality in u_{m-2}, first result, $A_{m-2} = C = 0$

At this step we assume the existence of two generic conserved densities $\rho = Pu_{m-1}^2+Qu_{m-1}+R$ and $\eta = Su_m^2 + Tu_m + U$ and we use the relation $S = F_m^{2/m}P$ given in Remark 3.5 and obtain $CP = AP_{m-2} = APa_{m-2}/a = 0$. Then we compute the explicit form of the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$ to prove that these imply that $A_{m-2} = C = 0$.

Proposition 4.4.1 *Let* $u_t = Au_m + Cu_{m-1}^2 + Du_{m-1} + E$, $m \geq 19$, $m = 2k+1$ *with* $|A| = |C| = |D| = |E| = m-2$, *and assume that it admits two conserved densities*

$$\rho = Pu_{m-1}^2 + Qu_{m-1} + R, \tag{4.22}$$

$$\eta = Su_m^2 + Tu_m + U, \tag{4.23}$$

where $|P| = |Q| = |R| = m-2$, $|T| = |U| = m-1$ *and* $S = F_m^{2/m} P$. *Then*

$$CP = AP_{m-2} = APa_{m-2}/a = 0. \tag{4.24}$$

Proof. If we compute (3.2) and (3.3) for $l=-1$ $F_m = A = a^m$, $A_{m-2} = ma^{m-1}a_{m-2} = (2k+1)a^{m-1}a_{m-2}$ and $\rho_{n,n} = \rho_{m-1,m-1} = 2P$ we obtain the coefficient of u_{m-1} in (3.2) as

$$4CP - (2k+1)\,AP_{m-2} + (2k-1)(2k+1)AP\frac{a_{m-2}}{a} = 0 \tag{4.25}$$

and the coefficient of u_{m+1} in (3.3) as

$$\begin{aligned}&12k^2CP - \left(2k^3+3k^2+13k+6\right)AP_{m-2}\\ &+\left(2k^3-3k^2+13k+6\right)(2k+1)AP\frac{a_{m-2}}{a} = 0.\end{aligned} \tag{4.26}$$

Then for $l=0$ $F_m = A = a^m$ and $\rho_{n,n} = \rho_{m,m} = 2S = 2a^2P$ the coefficient of u_{m+1} in (3.3) is obtained as

$$\begin{aligned}&12(k^2+2k+1)PC - (2k^3+3k^2+25k+12)AP_{m-2}+\\ &+(4k^4+4k^3+23k^2-19k-6)AP\frac{a_{m-2}}{a} = 0\end{aligned} \tag{4.27}$$

Equations (4.25), (4.26) and (4.27) form the following system:

$$\begin{bmatrix} 4 & -(2k+1) & (4k^2-1) \\ 12k^2 & -(2k^3+3k^2+13k+6) & (4k^4-4k^3+23k^2+25k+6) \\ 12(k^2+2k+1) & -(2k^3+3k^2+25k+12) & (4k^4+4k^3+23k^2-19k-6) \end{bmatrix} \begin{bmatrix} CP \\ AP_{m-2} \\ AP\frac{a_{m-2}}{a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.28}$$

The coefficient matrix in this linear system is singular for $k = -\frac{1}{2}, -\frac{3}{2}, 2$. Notice that for $k=2$, $m=5$ is also an exception. For $k \neq 2$, the system has only the trivial solution. Thus from (4.28) we have (4.24)

$\square$

Now we shall compute the canonical densities $\rho^{(1)}$, $\rho^{(2)}$, and $\rho^{(3)}$ and we shall see that their conservation will give $A_{m-2} = C = 0$.

By substituting $u_t = Au_m + Cu_{m-1}^2 + Du_{m-1} + E$, $m \geq 19$, $m = 2k+1$ with $A = a^m$, $|A| = |C| = |D| = |E| = m-2$, in the canonical densities $\rho^{(1)}$, $\rho^{(2)}$, and $\rho^{(3)}$ one can see that they reduce to

$$\begin{aligned}
\rho^{(1)} &= Pu_{m-1}^2 + Qu_{m-1} + R, \\
\rho^{(2)} &= Su_{m-1}^3 + Tu_{m-1}^2 + Vu_{m-1} + U, \\
\rho^{(3)} &= Wu_m^2 + Yu_m + Z,
\end{aligned}$$

with $|Q| = |R| = |T| = |V| = |U| = m-2$, $|Y| = |Z| = m-1$, and

$$\begin{aligned}
P &= a'' + \frac{m^2-1}{24}\frac{(a')^2}{a} + \frac{(1-m)}{m}\frac{C}{a^m}a' \\
&- \frac{1}{m}\frac{C'}{a^m}a + \frac{2(m-1)}{m^2}\frac{C^2}{a^{2m}}a
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
S &= a''a' - \frac{1}{6}\frac{(5+m)}{m}\frac{C}{a^m}a''a \\
&+ \frac{1}{6}\frac{(m-13)}{m}\frac{C'}{a^m}a'a + 2\frac{(m-1)}{m^2}\frac{C^2}{a^{2m}}a'a \\
&+ \frac{1}{6}\frac{(6m-m^2-5)}{m}\frac{C}{a^m}(a')^2 + \frac{2}{m^2}\frac{C'C}{a^{2m}}a^2 \\
&+ \frac{8}{3}\frac{(1-m)}{m^3}\frac{C^3}{a^{3m}}a^2
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
W &= a(a')^2 - 120\frac{m+3}{m(m^3+3m^2-121m+597)}\frac{C}{a^m}a'a^2 \\
&+ 240\frac{m-1}{m^2(m^3+3m^2-121m+597)}\frac{C^2}{a^{2m}}a^3
\end{aligned} \tag{4.31}$$

where $' = \frac{\partial}{\partial u_{m-2}}$. Note that as higher order conserved densities should be quadratic (Corollary 3.4) and top coefficients of conserved densities of consecutive orders are related (Remark 3.5), we have

$$S = 0 \qquad \text{and} \qquad W = a^2 P. \tag{4.32}$$

We will now prove that, Proposition 4.4.1 together with the conservation requirements of the $\rho^{(i)}$'s above will imply $A_{m-1} = C = 0$.

Corollary 4.4.2 *Let* $u_t = Au_m + Cu_{m-1}^2 + Du_{m-1} + E$, $m \geq 19$, $m = 2k+1$ *with* $A = a^m$, $|A| = |C| = |D| = |E| = m-2$. *Then if* $\rho^{(1)}$, $\rho^{(2)}$, *and* $\rho^{(3)}$ *are conserved quantities, then*

$$A_{m-2} = C = 0. \tag{4.33}$$

Proof. Let's first consider the case where $C = 0$. In this case, P reduces to $P = (a')^2/a$, and substituting this in (4.24), from $APa'/a = 0$ we get $a' = 0$, hence $A_{m-2} = 0$.

If $C \neq 0$, from (4.24), $CP = 0$ implies $P = 0$. But as $W = a^2 P$, we have $W = 0$ also. Now, $(4.29, 4.30, 4.31)$, is a system of three nonlinear differential equations for a and C. We can in principle solve a' from $W = 0$ in terms of C as the root of a quadratic equation, then substitute in $P = 0$ and $S = 0$ to get a system for C and C'. With symbolic computations it is possible to see that C is zero. We give below an analytic proof of this fact.

We define $\tilde{a} = a'/a$ and $\tilde{C} = C/a^m$. Then $C'/a^m = \tilde{C}' + m\tilde{a}\tilde{C}$ and $a''/a = \tilde{a}' + \tilde{a}^2$. Hence

$$\begin{aligned} \frac{P}{a} &= \frac{m^2+23}{24}\tilde{a}^2 + \frac{1-2m}{m}\tilde{a}\tilde{C} + \tilde{a}' \\ &+ \frac{2(m-1)}{m^2}\tilde{C}^2 - \frac{1}{m}\tilde{C}', \end{aligned} \tag{4.34}$$

$$\frac{S}{a^2} = \tilde{a}^3 + \frac{1}{6}\frac{(5m-m^2-10)}{m}\tilde{a}^2\tilde{C}$$

$$
\begin{aligned}
&+ \quad \frac{4m-2}{m^2}\tilde{a}\tilde{C}^2 + \frac{1}{6}\frac{(m-13)}{m}\tilde{a}\tilde{C} \\
&- \quad \frac{1}{6}\frac{(5+m)}{m}\tilde{a}'\tilde{C} + \tilde{a}\tilde{a}' + \frac{2}{m^2}\tilde{C}\tilde{C}' \\
&+ \quad \frac{8}{3}\frac{(1-m)}{m^3}\tilde{C}^3, \qquad (4.35)
\end{aligned}
$$

$$
\begin{aligned}
\frac{W}{a^3} &= \tilde{a}^2 - 120\frac{m+3}{m(m^3+3m^2-121m+597)}\tilde{a}\tilde{C} \\
&+ 240\frac{m-1}{m^2(m^3+3m^2-121m+597}\tilde{C}^2. \qquad (4.36)
\end{aligned}
$$

Note that W/a^3 is of the form

$$
\frac{W}{a^3} = \tilde{a}^2 - 2\kappa_1\tilde{C}\tilde{a} + \kappa_2\tilde{C}^2 \qquad (4.37)
$$

where κ_1 and κ_2 are constants that can be identified from (4.36) and since $W = 0$

$$
\tilde{a}_{1,2} = \tilde{C}(-\kappa_1 \pm \sqrt{\kappa_1^2 - \kappa_2}). \qquad (4.38)
$$

If γ denotes the coefficient of C' in the expression above, we have $\tilde{a} = \gamma\tilde{C}$ and $\tilde{a}' = \gamma\tilde{C}'$ and we obtain

$$
\begin{aligned}
\frac{P}{a} &= \left(\frac{m^2-23}{24}\gamma^2 + \frac{1-2m}{m}\gamma + 2\frac{m-1}{m^2}\right)\tilde{C}^2 \\
&+ \left(\gamma - \frac{1}{m}\right)\tilde{C}' = 0, \qquad (4.39)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}^{-1}\frac{S}{a^2} &= \left(\gamma^3 - \frac{\gamma^2}{3}\ \frac{4m+5}{m} + \gamma\ \frac{4m-2}{m^2} + \frac{8}{3}\ \frac{1-m}{m^3}\right)\tilde{C}^2 \\
&+ \left(\gamma^2 - \frac{3}{m}\gamma + \frac{2}{m^2}\right)\tilde{C}' = 0. \qquad (4.40)
\end{aligned}
$$

Now (4.39) and (4.40) form a system of homogeneous linear system for $\tilde{C}'$ and C^2. It can be checked that the coefficient matrix is nonsingular, hence $\tilde{C} = 0$. From (4.38), we have $\tilde{a} = a' = 0$, and the proof is complete. □

4.5 Step 5. Polinomiality in u_{m-2}, second result: $D_{m-2,m-2}=0$

We will now prove that D is linear in u_{m-2}

Proposition 4.5.1 *Let*

$$u_t = Au_m + Du_{m-1} + E,\ \ m \geq 19, \tag{4.41}$$

with $|A| = m-3,\ \ |D| = |E| = m-2,$ *and assume that it admits a conserved density of the form*

$$\rho = Pu_{m-1}^2 + Qu_{m-1} + R,, \tag{4.42}$$

where $|P| = m-3,\ |Q| = |R| = m-2.$ *Then*

$$D_{m-2,m-2} = 0. \tag{4.43}$$

Proof. We compute (3.2) for equations (4.41) and (4.42) with $l=-1,\ F_m = A,\ \ \rho_{n,n} = 2P$ to get:

$$\left[k+\frac{1}{2}\right]2P_{m-3}Au_{m-2} - \left[k-\frac{1}{2}\right]2PA_{m-3}u_{m-2} = 2PD \tag{4.44}$$

Differentiating (4.44) twice with respect to u_{m-2} we obtain

$$PD_{m-2,m-2} = 0 \tag{4.45}$$

□

It can be checked that the canonical density $\rho^{(3)}$ in $(A.5)$ is of the form (4.42) and the coefficient P of the quadratic top derivative u_{m-1}^2, is as follows:

$$\begin{aligned} P &= \frac{m^4 - 10m^2 + 720m - 2151}{m^4 - 10m^2 + 9} a a_{m-3}^2 \\ &+ \frac{60(-3m^2 + 8m + 3)}{m(m^4 - 10m^2 + 9)a^m} a^2 a_{m-3} D_{m-2} \\ &+ \frac{60}{m^2(m^2+4m+3)a^{2m}} a^3 D_{m-2}^2 \end{aligned} \tag{4.46}$$

That is

$$P = P^{(0)} + P^{(1)}D' + P^{(2)}D'^2 \tag{4.47}$$

Hence $PD'' = 0$ implies

$$D''(P^{(0)} + P^{(1)}D' + P^{(2)}D'^2) = 0 \tag{4.48}$$

From this it follows that

$$D'' = D_{m-2,m-2} = 0 \tag{4.49}$$

Corollary 4.5.2 *If the canonical density $\rho^{(3)}$ is conserved D is linear in u_{m-2}.*

Thus we conclude that an integrable evolution equation should be of the form

$$u_t = Au_m + Gu_{m-1}u_{m-2} + Hu_{m-1} + E, \tag{4.50}$$

where $|A| = |G| = |H| = m-3$ and $|E| = m-2$.

4.6 Step 6. Polinomiality in u_{m-2}, third result: $E_{m-2,m-2,m-2,m-2} = 0$.

The final step is to determine the u_{m-2} dependency in E using the explicit form of $\rho^{(1)}$. For

$$u_t = Au_m + Gu_{m-1}u_{m-2} + Hu_{m-1} + E, \;\; m \geq 19, \tag{4.51}$$

with $|A| = |G| = |H| = m-3$ and $|E| = m-2$, the canonical density $\rho^{(1)}$ is of the form

$$\begin{aligned}
\rho^{(1)} &= \frac{u_{m-2}^2}{m^2-1}\left[\frac{a_{m-3}^2}{a}(m^2-1) + 36\frac{a_{m-3}G}{a^m}(\frac{1}{m}-1) + 24\frac{aG_{m-3}}{ma^m} + 12\frac{aG^2}{ma^{2m}}(1-\frac{1}{m})\right] \\
&+ \frac{2u_{m-2}u_{m-3}}{m^2-1}\left[\frac{a_{m-3}a_{m-4}}{a}(m^2-1) + 18\frac{a_{m-4}G}{a^m}(\frac{1}{m}-1) + 12\frac{G_{m-4}a}{ma^m}\right] \\
&+ 12u_{m-2}\frac{H}{a^{2m}m^2(m+1)}\left[2aG - ma^m a_{m-3}\right] \\
&+ u_{m-3}^2\frac{a_{m-4}^2}{a} - 12u_{m-3}\frac{a_{m-4}H}{m(m+1)a^m} \\
&+ 12\frac{a}{m^2(m^2-1)a^{2m}}\left[-2ma^m E_{m-2} + H^2(m-1)\right]
\end{aligned} \tag{4.52}$$

where $a^m = A$

Proposition 4.6.1 *Let*

$$u_t = Au_m + Gu_{m-1}u_{m-2} + Hu_{m-1} + E,\ \ m \geq 19, \tag{4.53}$$

with $|A| = |G| = |H| = m-3$ *and* $|E| = m-2$*, Then if* $\rho^{(1)}$ *is a conserved quantity, then*

$$E_{m-2,m-2,m-2,m-2} = 0. \tag{4.54}$$

Proof. Notice that $\rho^{(1)}$ has order $m-2$ but it is not a priori even polynomial in u_{m-2} since we don't know the form of E. We substitute (4.53) in (3.2) with $l=-2,\ \ F_m = A$ and

$F_{m-1} = Gu_{m-2} + H$

$$(k+\frac{1}{2})AD\rho^{(1)}_{m-2,m-2} - (k-\frac{3}{2})DA\rho^{(1)}_{m-2,m-2} = (Gu_{m-2}+H)\rho^{(1)}_{m-2,m-2} \tag{4.55}$$

The coefficient of u_{m-1} in (4.55) is:

$$\rho^{(1)}_{m-2,m-2,m-2} = 0, \tag{4.56}$$

hence since $a \neq 0$, it follows that

$$\frac{\partial^3 \rho^{(1)}}{\partial {u_{m-2}}^3} = -\frac{24a^{-m+1}}{m(m^2-1)}E_{m-2,m-2,m-2,m-2} = 0. \tag{4.57}$$

□

Result C: Polynomiality in u_m, u_{m-1}, u_{m-2}

Corollary 4.6.2 *If the canonical densities* $\rho^{(1)}$*,* $\rho^{(2)}$ *and* $\rho^{(3)}$ *are conserved quantities for the evolution equation* $u_t = F$*, then*

$$F = Au_m + Gu_{m-2}u_{m-1} + Hu_{m-1} + Ju^3_{m-2} + Lu^2_{m-2} + Nu_{m-2} + S \tag{4.58}$$

where A*,* G*,* H*,* J*,* L*,* N*,* S *are functions of* $x, t, u, \ldots u_{m-3}$.

Remark 4.7 By the use of the conserved densities $\rho^{(i)}, \quad i = 1, 2, 3$ we have shown that A is independent of u_{m-3} and $G = 0$. However it was not possible to obtain polynomiality in u_{m-3} because further conserved densities were needed.

5. CONCLUSION

In the present paper we obtained all the polynomiality information that could be extracted from the conserved densities up to $\rho^{(3)}$ (See Remark 4.7). In general computations we haven't used $\rho^{(-1)}$ and $\rho^{(0)}$, but using these for $m = 7$ didn't give any further information.

For $m = 7$, in initial computations we used dependencies in all derivatives. But at later stages this was impossible and in the search of computationally efficient methods, we noticed that all polynomiality results followed from the coefficient of the top order term in (3.2) and (3.3). The use of only the top dependency lead us to formulate a graded algebra structure on the polynomials in the derivatives u_{k+i}, that we called "level grading" [8]. This structure is based on the fact that derivatives of a function of $x, t, u, \ldots, u_k$ are polynomial in higher order derivatives and have a natural scaling by the order of differentiation above the "base level k." The crucial point is that equations relevant for obtaining polynomiality results involve only the terms with top scaling weight with respect to level grading. Thus one can work with the dependency on the top level term only and reduce the scope of symbolic computations to a feasible range. Applications of the "level grading" structure to the classification problem will be discussed elsewhere.

By the remark following Proposition 3.1, the polynomiality results obtained in Section 4 are valid for $m \geq 19$. For $m = 7, 9, 11, 13, 15, 17$ the conserved density conditions are computed, directly, *without using* (3.2) and (3.3), with the symbolic programming language REDUCE and

it is shown that the evolution equation have again the form given by (4.58). The results of Corollary 4.6.2 are valid for all $m \geq 7$ and we restate it here for convenience.

Corollary 4.6.2 *Let*

$$u_t = F(x, t, u, \ldots, u_m)$$

be a scalar evolution equation in one space dimension of order $m = 2k + 1$ *where* $m \geq 7$*. If the canonical densities* $\rho^{(i)}$*,* $i = 1, 2, 3$ *given in the Appendix are conserved quantities then*

$$F = Au_m + Gu_{m-2}u_{m-1} + Hu_{m-1} + Ju_{m-2}^3 + Lu_{m-2}^2 + Nu_{m-2} + S.$$

where A, G, H, J, L, N, S are functions of $x, t, u, \ldots u_{m-3}$*.*

For future work, the graded algebra structure seems to be promising tool both for the computation of conserved densities and for obtaining integrability conditions. In fact, conserved densities for seventh order equations have been computed and it has been observed that there are candidates for integrable equations where A may depend on u_4 and is non-polynomial in u_4. In this sense, Corollary 4.6.2 is the best polynomiality result for evolution equations of order 7.

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APPENDIX A: Canonical Densities of the Formal Symmetry Method

If the evolution equation $u_t = F[u]$ is integrable, it is known that the quantities

$$\rho^{(-1)} = F_m^{-1/m}, \quad \rho^{(0)} = F_{m-1}/F_m, \tag{A.1a}$$

where

$$F_m = \frac{\partial F}{\partial u_m}, \quad F_{m-1} = \frac{\partial F}{\partial u_{m-1}} \tag{A.1b}$$

are conserved densities for equations of any order [3]. The next three canonical densities *up to total derivatives* computed in [5] are presented below. The expressions of the canonical densities

up to total derivatives are convenient for the present work but one has to be careful in general in adding a total derivative to the canonical densities. Because if $\rho^{(i)}$ is a canonical density with $D_t\rho^{(i)} = D\sigma^{(i)}$, then $\rho^{(i+k)}$ will involve $\sigma^{(i)}$ after some k [3]. Hence although total derivatives in $\rho^{(i)}$ are irrelevant for the existence of $\sigma^{(i)}$, they should not be omitted whenever $\sigma^{(i)}$ enters in the expression of another canonical density. We shall use the following notation

$$a = F_m^{1/m}, \quad \alpha_{(i)} = \frac{F_{m-i}}{F_m}, \quad i = 1,2,3,4. \tag{A.2}$$

$$\rho^{(1)} = a^{-1}(Da)^2 - \frac{12}{m(m+1)}Da\alpha_{(1)} + a\left[\frac{12}{m^2(m+1)}\alpha_{(1)}^2 - \frac{24}{m(m^2-1)}\alpha_{(2)}\right], \tag{A.3}$$

$$\begin{aligned}\rho^{(2)} &= a(Da)\left[D\alpha_{(1)} + \frac{3}{m}\alpha_{(1)}^2 - \frac{6}{(m-1)}\alpha_{(2)}\right] \\ &+ 2a^2\left[-\frac{1}{m^2}\alpha_{(1)}^3 + \frac{3}{m(m-1)}\alpha_{(1)}\alpha_{(2)} - \frac{3}{(m-1)(m-2)}\alpha_{(3)}\right],\end{aligned} \tag{A.4}$$

$$\begin{aligned}\rho^{(3)} &= a(D^2a)^2 - \frac{60}{m(m+1)(m+3)}a^2D^2aD\alpha_{(1)} + \frac{1}{4}a^{-1}(Da)^4 \\ &+ 30a(Da)^2\left[\frac{(m-1)}{m(m+1)(m+3)}D\alpha_{(1)} + \frac{1}{m^2(m+1)}\alpha_{(1)}^2 - \frac{2}{m(m^2-1)}\alpha_{(2)}\right] \\ &+ \frac{120}{m(m^2-1)(m+3)}a^2Da\left[-\frac{(m-1)(m-3)}{m}\alpha_{(1)}D\alpha_{(1)} + (m-3)D\alpha_{(2)}\right. \\ &- \left.\frac{(m-1)(2m-3)}{m^2}\alpha_{(1)}^3 + \frac{6(m-2)}{m}\alpha_{(1)}\alpha_{(2)} - 6\alpha_{(3)}\right] \\ &+ \frac{60}{m(m^2-1)(m+3)}a^3\left[\frac{(m-1)}{m}(D\alpha_{(1)})^2 - \frac{4}{m}D\alpha_{(1)}\alpha_{(2)} + \frac{(m-1)(2m-3)}{m^3}\alpha_{(1)}^4\right. \\ &- \left.4\frac{(2m-3)}{m^2}\alpha_{(1)}^2\alpha_{(2)} + \frac{8}{m}\alpha_{(1)}\alpha_{(3)} + \frac{4}{m}\alpha_{(2)}^2 - \frac{8}{(m-3)}\alpha_{(4)}\right].\end{aligned} \tag{A.5}$$

Evolution equation of order m with generic and canonical densities

Step	Evolution Equation	Type of the conserved density	Form of the conserved density	Result
1	$u_t = F[u]$	generic	$\rho = Pu_{m+1}^2 + Qu_{m+1} + R$	$\frac{\partial^2 F}{\partial {u_m}^2} = 0$
2	$u_t = Au_m + B$	generic	$\rho = Pu_m^2 + Qu_m + R$	$\frac{\partial A}{\partial u_{m-1}} = 0$
3	$u_t = Au_m + B$	canonical	$\rho = \rho^{(3)}$	$\frac{\partial^3 B}{\partial {u_{m-1}}^3} = 0$
4	$u_t = Au_m + Cu_{m-1}^2$ $+Du_{m-1} + E$	generic and canonical	$\rho = Pu_{m-1}^2 + Qu_{m-1} + R$ $\eta = Su_m^2 + Tu_m + U$	$\frac{\partial A}{\partial u_{m-2}} = 0,$ $C = 0$
5	$u_t = Au_m + Du_{m-1}$ $+E$	generic and canonical	$\rho = Pu_{m-1}^2 + Qu_{m-1} + R$	$\frac{\partial^2 D}{\partial {u_{m-2}}^2} = 0$
6	$u_t = Au_m$ $+Gu_{m-1}u_{m-2}$ $+Hu_{m-1} + E$	canonical	$\rho = \rho^{(1)}$	$\frac{\partial^4 E}{\partial u_{m-2}^4} = 0$

Table1:Polynomiality results for the general case

APPENDIX B: Expressions of k'th Order Derivatives[5]

$$D^k\varphi = \varphi_n u_{n+k} + P(u_{n+k-1}), \quad k \geq 1 \tag{A.6a}$$

$$D^k\varphi = \varphi_n u_{n+k} + [\varphi_{n-1} + kD\varphi_n]\, u_{n+k-1} + P(u_{n+k-2}), \quad k \geq 3 \tag{A.6b}$$

$$\begin{aligned} D^k\varphi &= \varphi_n u_{n+k} + [\varphi_{n-1} + kD\varphi_n]\, u_{n+k-1} \\ &+ \left[\varphi_{n-2} + kD\varphi_{n-1} + \binom{k}{2} D^2\varphi_n\right] u_{n+k-2} + P(u_{n+k-3}), \quad k \geq 5 \end{aligned} \tag{A.6c}$$

$$\begin{aligned} D^k\varphi &= \varphi_n u_{n+k} + [\varphi_{n-1} + kD\varphi_n]\, u_{n+k-1} \\ &+ \left[\varphi_{n-2} + kD\varphi_{n-1} + \binom{k}{2} D^2\varphi_n\right] u_{n+k-2} \\ &+ \left[\varphi_{n-3} + kD\varphi_{n-2} + \binom{k}{2} D^2\varphi_{n-1} + \binom{k}{3} D^3\varphi_n\right] u_{n+k-3} \\ &+ P(u_{n+k-4}), \quad k \geq 7 \end{aligned} \tag{A.6d}$$

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